

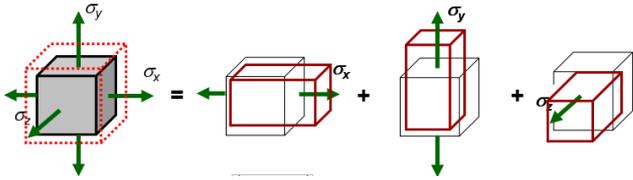


Finite element method (FEM1)

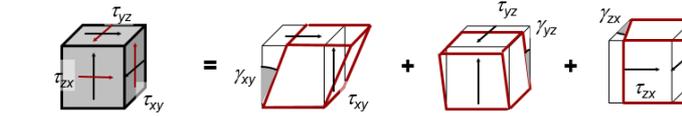
Lecture 3A. Two dimensional (2D) cases - CST element

03.2026

Hooke's law for an isotropic material in a three-dimensional state



$$\begin{aligned}\varepsilon_x &= \frac{1}{E} (\sigma_x - \nu(\sigma_y + \sigma_z)) \\ \varepsilon_y &= \frac{1}{E} (\sigma_y - \nu(\sigma_z + \sigma_x)) \\ \varepsilon_z &= \frac{1}{E} (\sigma_z - \nu(\sigma_x + \sigma_y))\end{aligned}$$



$$\begin{aligned}\gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{yz} &= \frac{\tau_{yz}}{G} \\ \gamma_{zx} &= \frac{\tau_{zx}}{G}\end{aligned}$$

E – Young modulus
 ν – Poisson ratio
 G – shear modulus

$$G = \frac{E}{2(1+\nu)}$$

vector of stress components:

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}_{6 \times 1}$$

vector of strain components:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}_{6 \times 1}$$

$$\{\sigma\} = [D] \{\varepsilon\}$$

$6 \times 1 \quad 6 \times 6 \quad 6 \times 1$

Constitutive matrix:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix}_{6 \times 6}$$

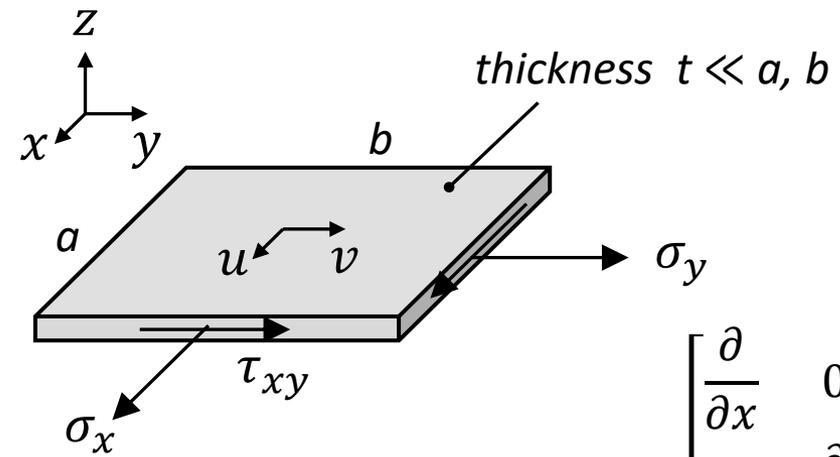
Plane stress (thin plates, shells)

$$\sigma_x ; \sigma_y ; \sigma_z = 0$$

$$\tau_{xy} ; \tau_{yz} = 0 ; \tau_{zx} = 0$$

$$\epsilon_x ; \epsilon_y ; \epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} ; \gamma_{yz} = 0 ; \gamma_{zx} = 0$$



displacement vector:

$$[u] = [u, v]$$

1 × 2

$$[R] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

3 × 2

Gradient matrix

$$\{\epsilon\} = [R]\{u\}$$

3 × 1 3 × 2 2 × 1

vector of strain components:

$$[\epsilon] = [\epsilon_x, \epsilon_y, \gamma_{xy}]$$

1 × 3

vector of stress components:

$$[\sigma] = [\sigma_x, \sigma_y, \tau_{xy}]$$

1 × 3

$$\{\sigma\} = [D]\{\epsilon\}$$

3 × 1 3 × 3 3 × 1

Constitutive matrix for Plain Stress:

$$[D] = \frac{E}{(1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1 - \nu) \end{bmatrix}$$

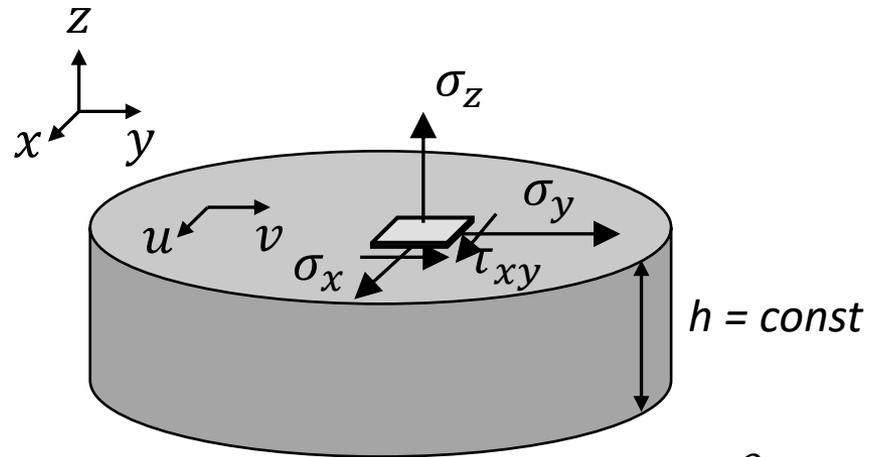
Plane strain (infinitely long pipe, prism and roller)

$$\sigma_x ; \sigma_y ; \sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\tau_{xy} ; \tau_{yz} = 0 ; \tau_{zx} = 0$$

$$\epsilon_x ; \epsilon_y ; \epsilon_z = 0$$

$$\gamma_{xy} ; \gamma_{yz} = 0 ; \gamma_{zx} = 0$$



displacement vector:

$$[u] = [u, v]$$

1 x 2

vector of strain components:

$$[\epsilon] = [\epsilon_x, \epsilon_y, \gamma_{xy}]$$

1 x 3

vector of stress components:

$$[\sigma] = [\sigma_x, \sigma_y, \tau_{xy}]$$

1 x 3

$$\{\epsilon\} = [R] \{u\}$$

3 x 1 3 x 2 2 x 1

$$\{\sigma\} = [D] \{\epsilon\}$$

3 x 1 3 x 3 3 x 1

Gradient matrix

$$[R] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

Constitutive matrix for plain strain:

$$[D] = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & \frac{1}{2}(1 - 2\nu) \end{bmatrix}$$

Axial symmetry (rotating disc)

$$\sigma_x \ ; \ \sigma_y \ ; \ \sigma_z$$

$$\tau_{xy} \ ; \ \tau_{yz} = 0 \ ; \ \tau_{zx} = 0$$

$$\varepsilon_x \ ; \ \varepsilon_y \ ; \ \varepsilon_z = 0$$

$$\gamma_{xy} \ ; \ \gamma_{yz} = 0 \ ; \ \gamma_{zx} = 0$$

displacement vector: $[u] = [u, v]$
1 × 2

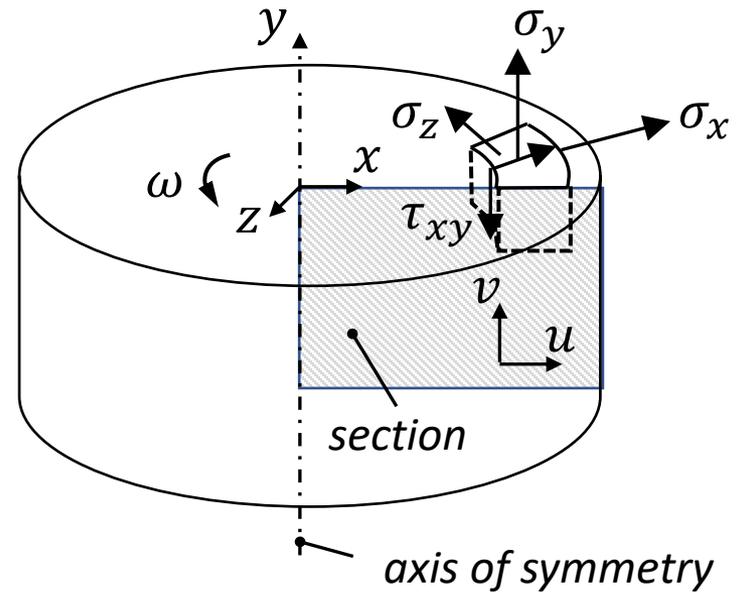
vector of strain components:

$$[\varepsilon] = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}]$$
1 × 4

vector of stress components:

$$[\sigma] = [\sigma_x, \sigma_y, \sigma_z, \tau_{xy}]$$
1 × 4

directions:
 x – radial
 y – longitudinal
 z – hoop



gradient matrix:

$$[R] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ \frac{1}{x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$
4 × 2

$$\{\varepsilon\} = [R] \{u\}$$
4 × 1 4 × 2 2 × 1

$$\{\sigma\} = [D] \{\varepsilon\}$$
4 × 1 4 × 4 4 × 1

$$[D] = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & \nu & 0 \\ \nu & 1 - \nu & \nu & 0 \\ \nu & \nu & 1 - \nu & 0 \\ 0 & 0 & 0 & 0.5 - \nu \end{bmatrix}$$

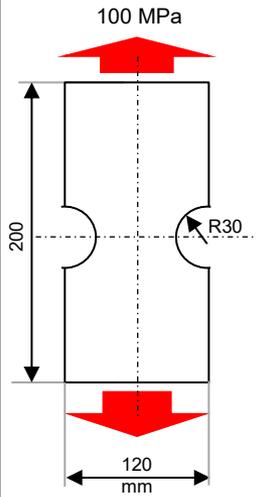
Constitutive matrix for axial symmetry

Example of using the 2D element option

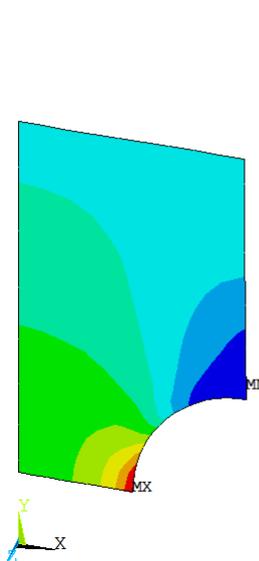
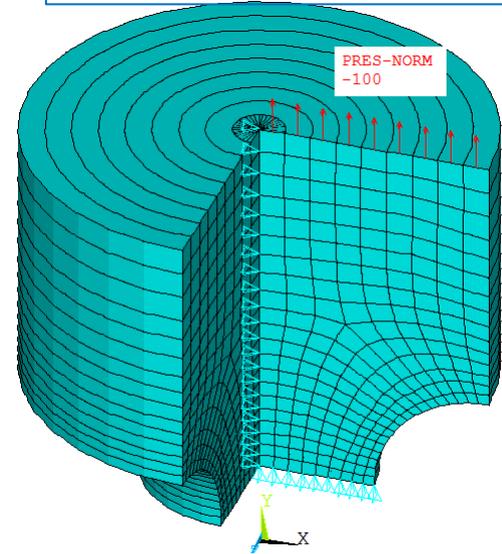
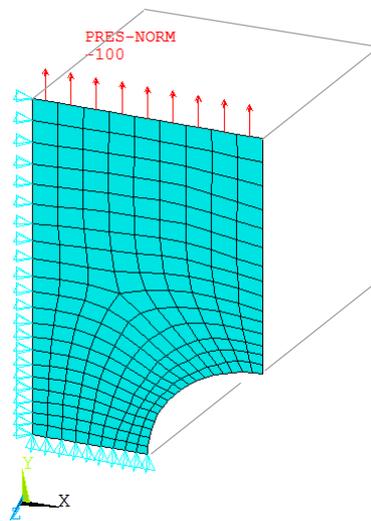
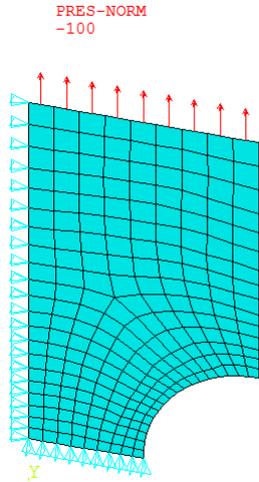
Plain stress

Plain strain

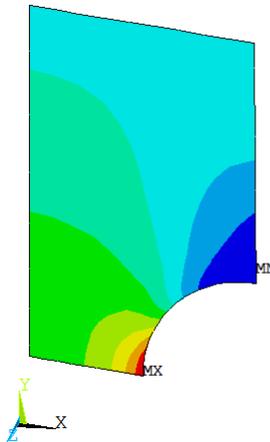
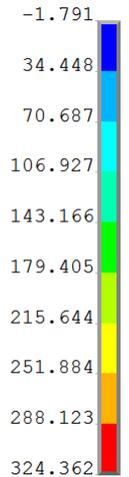
Axial symmetry



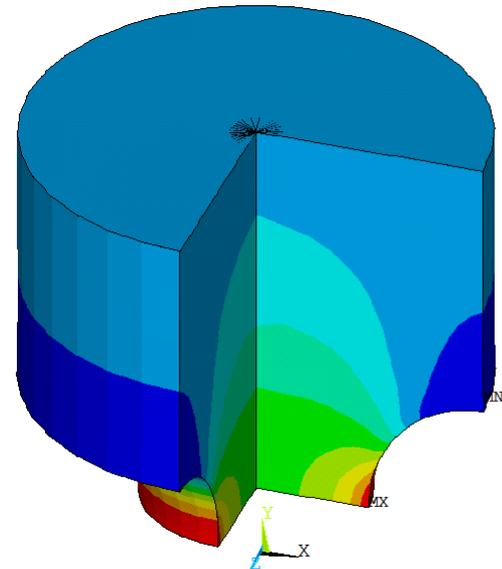
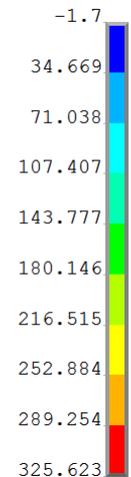
$E=2 \cdot 10^5 \text{ MPa}$
 $\nu = 0.3$



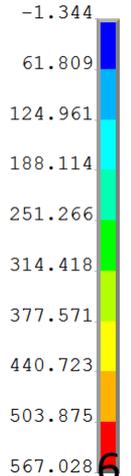
NODAL SOLUTION
 SY
 (AVG)



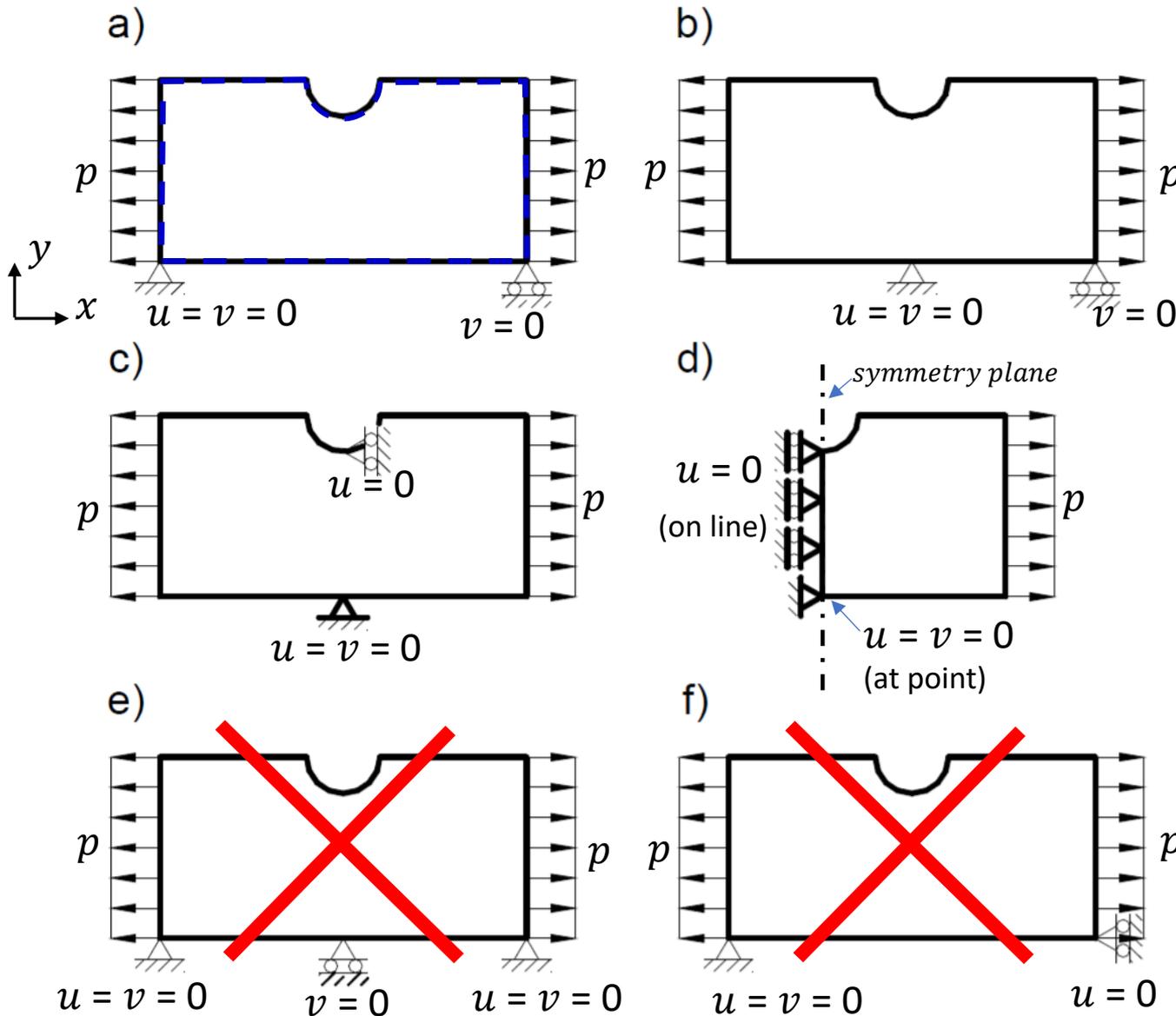
NODAL SOLUTION
 SY
 (AVG)



NODAL SOLUTION
 /EXPANDED
 SY
 (AVG)



Support conditions for a 2D plate loaded with forces in equilibrium

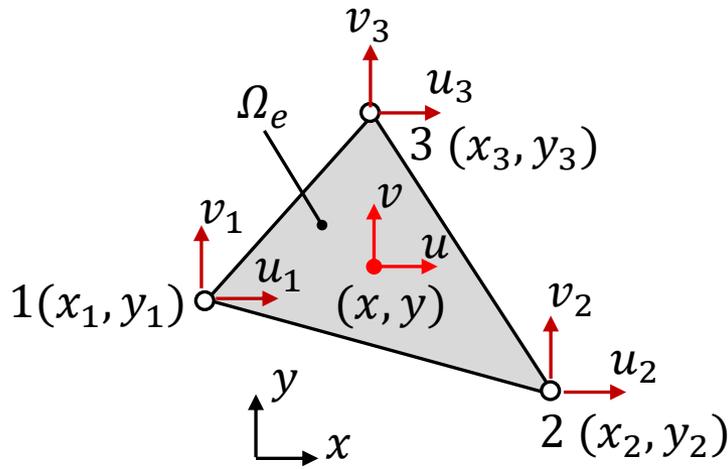


Even in the case of self-balanced loading, it is necessary to assign a number of degrees of freedom for the model that will prevent the possibility of movement as a rigid body.

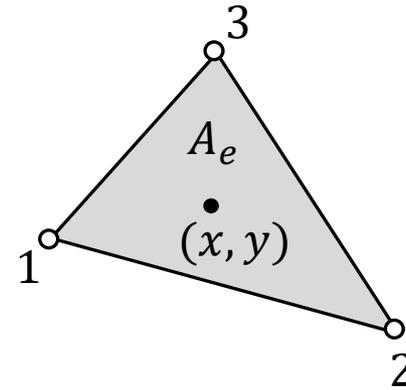
At the same time, we should not restrict the freedom of deformation so that the stiffness matrix does not become singular and the solution ambiguous.

Correct constraints:
 (constrained rigid body motion and correct deformation):
 a, b, c, d

CST (Constant Strain Triangle) finite element (2D, 3-node triangle)



$$n = 3 ; n_p = 2 \rightarrow n_e = n \cdot n_p = 6$$

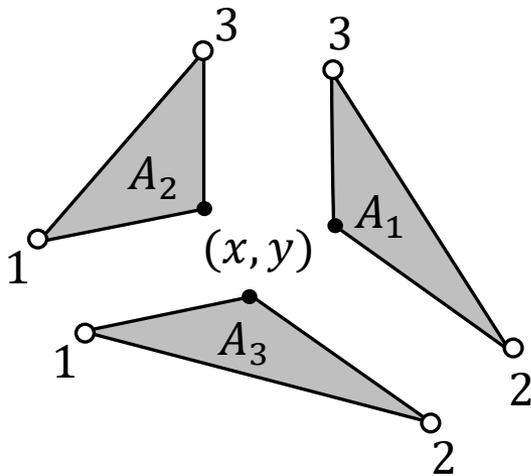


$$\int_{\Omega_e} d\Omega_e = A_e \cdot t_e$$

area
thickness

Area of a triangle with vertices 1,2,3:

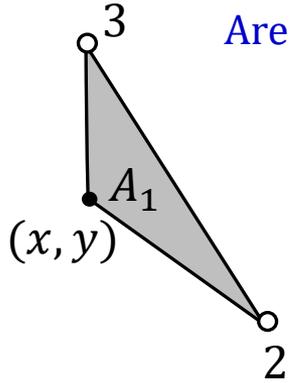
$$A_e = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \frac{x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1}{2}$$



$$A_e = A_1(x, y) + A_2(x, y) + A_3(x, y)$$

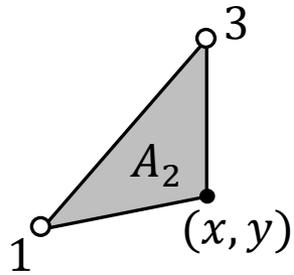
CST finite element

Area coordinates as functions of coordinates (x, y) :



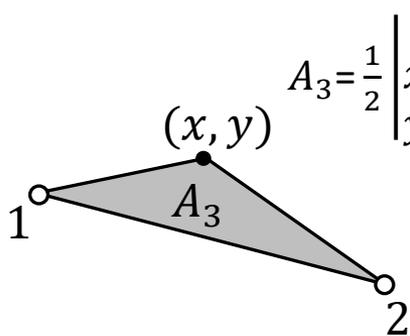
$$A_1 = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}$$

$$L_1 = \frac{A_1}{A_e} = \frac{1}{2A} \begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} = \frac{a_1 x_2 y_3 - x_3 y_2 + (y_2 - y_3)x + (x_3 - x_2)y}{2A}$$



$$A_2 = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix}$$

$$L_2 = \frac{A_2}{A_e} = \frac{1}{2A} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix} = \frac{a_2 x_3 y_1 - x_1 y_3 + (y_3 - y_1)x + (x_1 - x_3)y}{2A}$$



$$A_3 = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix}$$

$$L_3 = \frac{A_3}{A_e} = \frac{1}{2A} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix} = \frac{a_3 x_1 y_2 - x_2 y_1 + (y_1 - y_2)x + (x_2 - x_1)y}{2A}$$

$$A_e = A_1(x, y) + A_2(x, y) + A_3(x, y)$$

$$1 = L_1(x, y) + L_2(x, y) + L_3(x, y)$$

$L_i(x, y)$ - barycentric functions

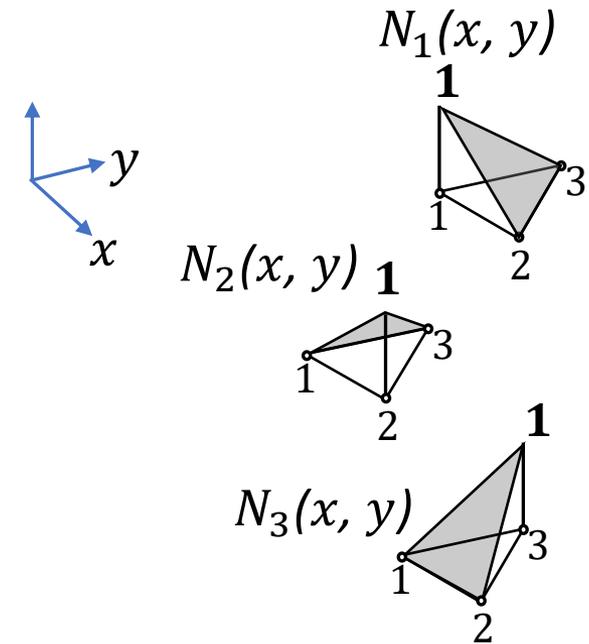
Shape functions of the CST element

shape functions = normalized area coordinates:

$$N_1(x, y) = L_1(x, y) = \frac{A_1(x, y)}{A_e} = \frac{1}{2A_e} (\mathbf{a}_1 + \mathbf{b}_1x + \mathbf{c}_1y)$$

$$N_2(x, y) = L_2(x, y) = \frac{A_2(x, y)}{A_e} = \frac{1}{2A_e} (\mathbf{a}_2 + \mathbf{b}_2x + \mathbf{c}_2y)$$

$$N_3(x, y) = L_3(x, y) = \frac{A_3(x, y)}{A_e} = \frac{1}{2A_e} (\mathbf{a}_3 + \mathbf{b}_3x + \mathbf{c}_3y)$$



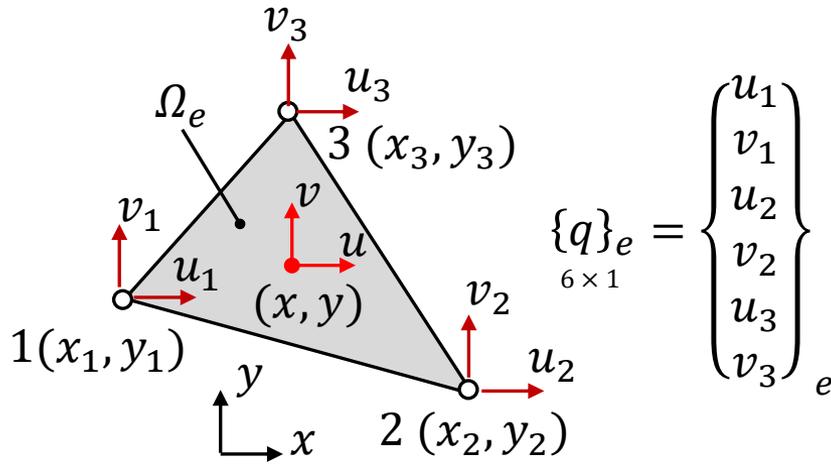
$$N_1(x, y) + N_2(x, y) + N_3(x, y) = 1$$

where:

$$\begin{aligned} \mathbf{a}_1 &= x_2y_3 - x_3y_2 & ; & & \mathbf{a}_2 &= x_3y_1 - x_1y_3 & ; & & \mathbf{a}_3 &= x_1y_2 - x_2y_1 \\ \mathbf{b}_1 &= y_2 - y_3 & ; & & \mathbf{b}_2 &= y_3 - y_1 & ; & & \mathbf{b}_3 &= y_1 - y_2 \\ \mathbf{c}_1 &= x_3 - x_2 & ; & & \mathbf{c}_2 &= x_1 - x_3 & ; & & \mathbf{c}_3 &= x_2 - x_1 \end{aligned}$$

node	$N_1(x, y)$	$N_2(x, y)$	$N_3(x, y)$
1	1	0	0
2	0	1	0
3	0	0	1

Isoparametric mapping in the CST element



vector of shape functions:

$$[N(x, y)]_{1 \times 3} = [N_1(x, y), N_2(x, y), N_3(x, y)]$$

vectors of nodal coordinates:

$$\{x_i\}_e = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{3 \times 1} \quad ; \quad \{y_i\}_e = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}_{3 \times 1}$$

coordinates of any point are based on shape functions and nodal parameters:

$$x = [N(x, y)]_{1 \times 3} \{x_i\}_e = N_1(x, y)x_1 + N_2(x, y)x_2 + N_3(x, y)x_3$$

$$y = [N(x, y)]_{1 \times 3} \{y_i\}_e = N_1(x, y)y_1 + N_2(x, y)y_2 + N_3(x, y)y_3$$

displacements at any point:

$$\{u(x, y)\}_{2 \times 1} = [N(x, y)]_{2 \times 6} \{q\}_e_{6 \times 1}$$

Isoparametric mapping - the same shape functions used for geometry and displacements

Strain-displacement matrix of the CST element

strain vector for plane stress or plane strain conditions:

$$\begin{aligned}
 \{\varepsilon\} &= [R] \{u\} = [R] [N] \{q\}_e = \\
 &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{bmatrix} \{q\}_e = \\
 &= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \{q\}_e = [B] \{q\}_e
 \end{aligned}$$

$$[B] = \frac{1}{2A_e} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \rightarrow \{\varepsilon\} = [B] \{q\}_e - \text{strain is constant}$$

$$\{\sigma\} = [D] \{\varepsilon\} - \text{stress is constant}$$

CST – Constant Strain Triangle

Elastic strain energy in the CST element. Local stiffness matrix

elastic strain energy in a finite element:

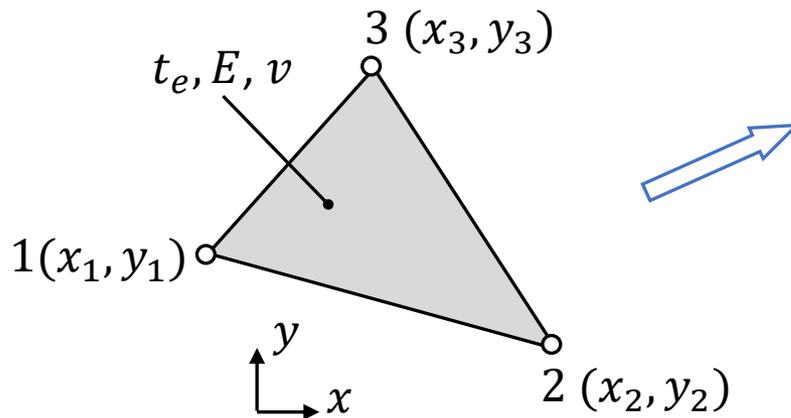
$$\begin{aligned}
 U_e &= \frac{1}{2} \int_{\Omega_e} [\varepsilon] \{\sigma\} d\Omega_e = \frac{1}{2} \int_{\Omega_e} \underset{1 \times 3}{[\varepsilon]} \underset{3 \times 1}{\{\sigma\}} d\Omega_e = \frac{1}{2} \underset{1 \times 6}{[q]_e} \underset{6 \times 3}{[B]^T} \underset{3 \times 3}{[D]} \underset{3 \times 6}{[B]} \underset{6 \times 1}{\{q\}_e} A_e t_e = \\
 &= \frac{1}{2} [q]_e [k]_e \{q\}_e
 \end{aligned}$$

$\underset{3 \times 1}{\{\sigma\}} = \underset{3 \times 3}{[D]} \underset{3 \times 1}{\{\varepsilon\}}$
 $\underset{1 \times 3}{[\varepsilon]} = \underset{1 \times 6}{[q]_e} \underset{6 \times 3}{[B]^T}$ $\underset{3 \times 1}{\{\varepsilon\}} = \underset{3 \times 6}{[B]} \underset{6 \times 1}{\{q\}_e}$

local stiffness matrix:

$$[k]_e = A_e t_e [B]^T [D] [B]$$

6×6 6×3 3×3 3×6



Potential energy of loading in the CST element

potential energy of loading in a finite element:

$$W_e = \int_{\Omega_e} [X] \{u\} d\Omega_e + \int_{\Gamma_{pe}} [p] \{u\} d\Gamma_{pe} =$$

$$\begin{matrix} \Omega_e & 1 \times 2 & 2 \times 1 \\ \Gamma_{pe} & 1 \times 2 & 2 \times 1 \end{matrix}$$

$$\{u\} = [N] \{q\}_e$$

$$\begin{matrix} 2 \times 1 & 2 \times 6 & 6 \times 1 \end{matrix}$$

$$= \int_{\Omega_e} [X][N] \{q\}_e d\Omega_e + \int_{\Gamma_{pe}} [p][N] \{q\}_e d\Gamma_{pe} =$$

$$\begin{matrix} \Omega_e & 1 \times 2 & 2 \times 6 & 6 \times 1 \\ \Gamma_{pe} & 1 \times 2 & 2 \times 6 & 6 \times 1 \end{matrix}$$

$$= \left(\int_{\Omega_e} [X][N] d\Omega_e + \int_{\Gamma_{pe}} [p][N] d\Gamma_{pe} \right) \{q\}_e = ([F^X]_e + [F^p]_e) \{q\}_e = [F]_e \{q\}_e$$

$$\begin{matrix} \Omega_e & 1 \times 2 & 2 \times 6 & 6 \times 1 \\ \Gamma_{pe} & 1 \times 2 & 2 \times 6 & 6 \times 1 \end{matrix} \quad \begin{matrix} 1 \times 6 & 1 \times 6 & 6 \times 1 & 1 \times 6 & 6 \times 1 \end{matrix}$$

equivalent load vector due to mass forces:

$$[F^X]_e = t_e \int_{A_e} [X][N] dA_e$$

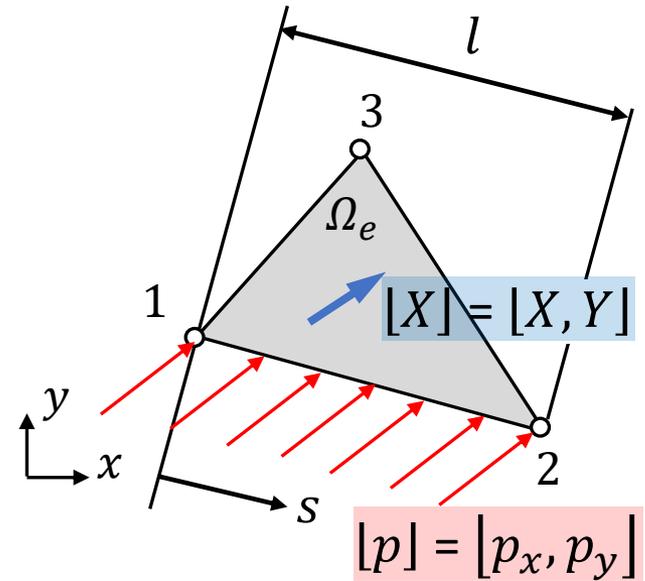
$$\begin{matrix} 1 \times 6 \\ A_e & 1 \times 2 & 2 \times 6 \end{matrix}$$

;

equivalent load vector due to surface load:

$$[F^p]_e = t_e \int_0^l [p][N] ds$$

$$\begin{matrix} 1 \times 6 \\ 0 & 1 \times 2 & 2 \times 6 \end{matrix}$$



Components of equivalent load vector in the CST element

$$[F^X]_e = t_e \int_{A_e} [X][N] dA_e$$

$$[F^p]_e = t_e \int_0^l [p][N] ds$$

equivalent load vector due to mass forces:

$$[F^X]_e = t_e \int_{A_e} [X, Y] \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} dA_e =$$

$$= t_e \int_{A_e} [XN_1, YN_1, XN_2, YN_2, XN_3, YN_3] dA_e = [F_{1e}^X, F_{2e}^X, F_{3e}^X, F_{4e}^X, F_{5e}^X, F_{6e}^X]$$

equivalent load vector due to surface load:

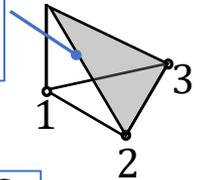
$$[F^p]_e = t_e \int_0^l [p_x, p_y] \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} ds =$$

$$= t_e \int_0^l [p_x, p_y] \begin{bmatrix} 1 - \frac{s}{l} & 0 & \frac{s}{l} & 0 & 0 & 0 \\ 0 & 1 - \frac{s}{l} & 0 & \frac{s}{l} & 0 & 0 \end{bmatrix} ds =$$

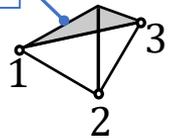
$$= t_e \int_0^l \left[p_x \left(1 - \frac{s}{l}\right), p_y \left(1 - \frac{s}{l}\right), p_x \frac{s}{l}, p_y \frac{s}{l}, 0, 0 \right] ds =$$

$$= [F_{1e}^p, F_{2e}^p, F_{3e}^p, F_{4e}^p, F_{5e}^p, F_{6e}^p]$$

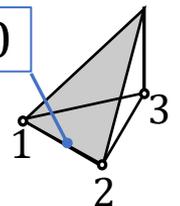
$$N_1(s)|_{1-2} = 1 - \frac{s}{l}$$



$$N_2(s)|_{1-2} = \frac{s}{l}$$



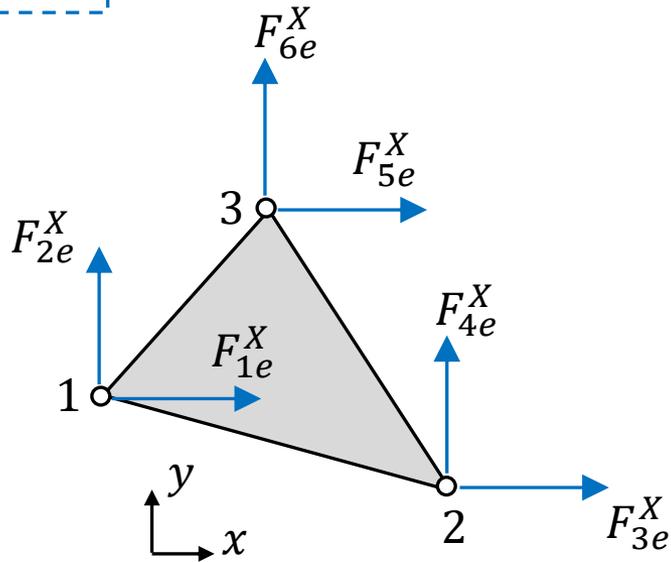
$$N_3(x, y)|_{1-2} = 0$$



Equivalent load vector in the CST element

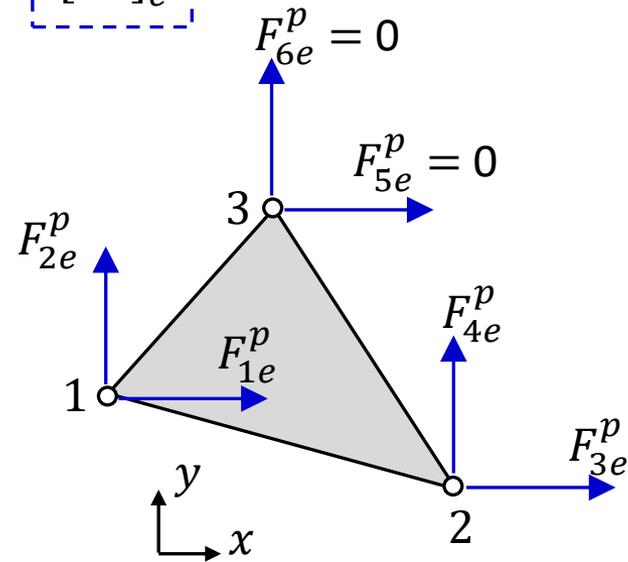
equivalent load vector
due to mass forces:

$$[F^X]_e$$



equivalent load vector
due to surface load:

$$[F^p]_e$$

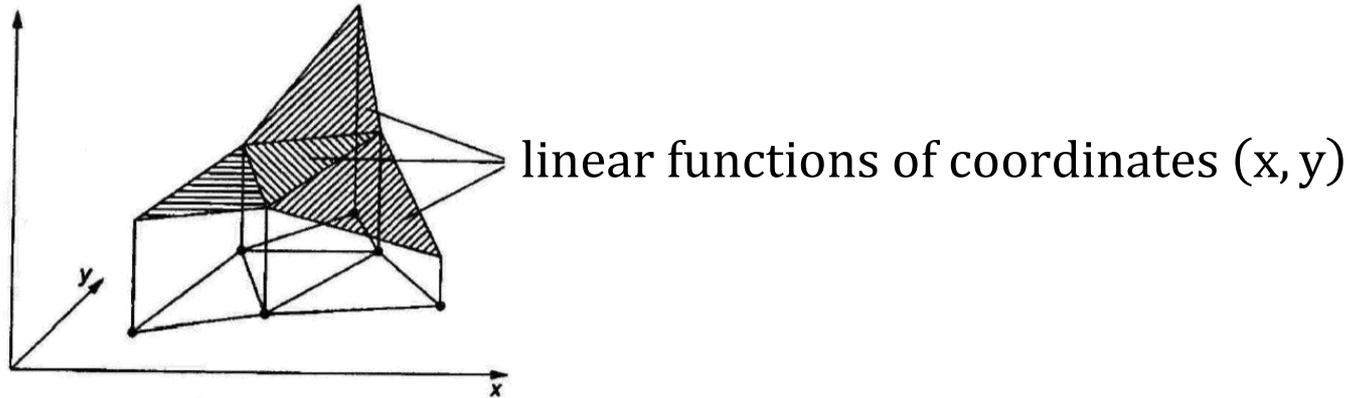


equivalent load vector:

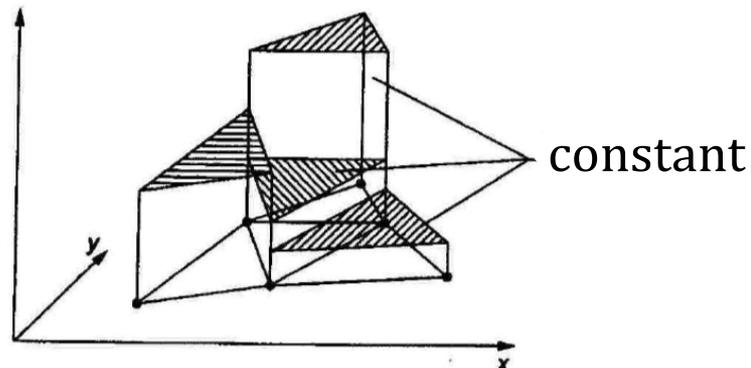
$$[F]_e = [F_{1e}^X + F_{1e}^p, F_{2e}^X + F_{2e}^p, F_{3e}^X + F_{3e}^p, F_{4e}^X + F_{4e}^p, F_{5e}^X + F_{5e}^p, F_{6e}^X + F_{6e}^p]$$

Results in the CST element

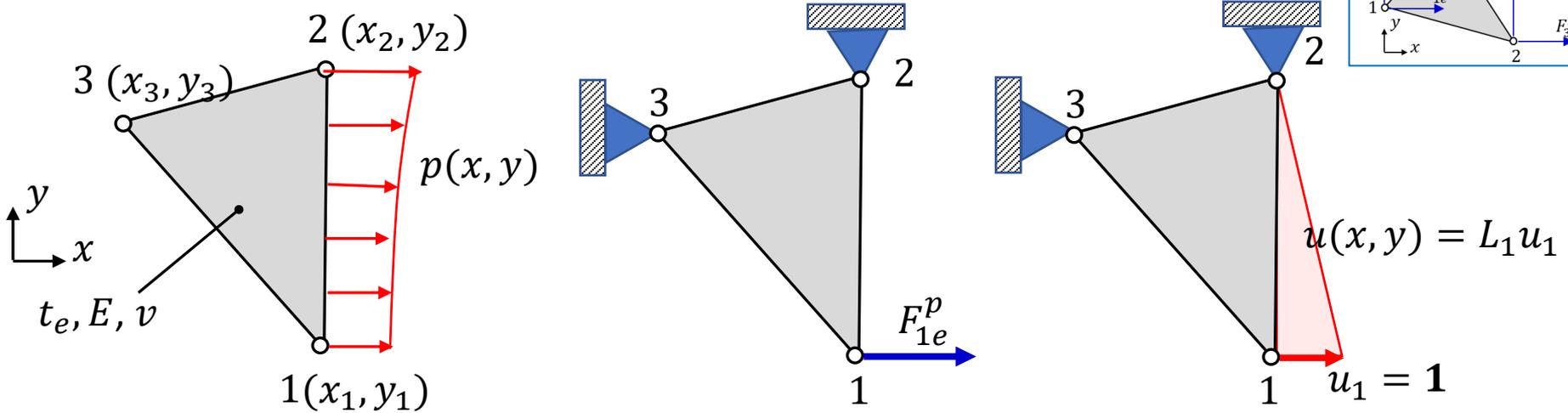
DOF solution : $u(x, y), v(x, y)$



element solution: $\{\sigma\}, \{\varepsilon\}$
 $3 \times 1 \quad 3 \times 1$



Example 1: Determination of the equivalent force in the CST element due to surface forces



equivalent load vector due to surface load:

$$[F^p]_e = t_e \int_0^l [p][N] ds$$

The work of the equivalent force F_{1e}^p on displacement 1

The work of load $p(x,y)$ on displacement $u(x,y)$

$$F_{1e}^p \cdot 1 = t_e \int_0^l p(x,y) u(x,y) dy$$

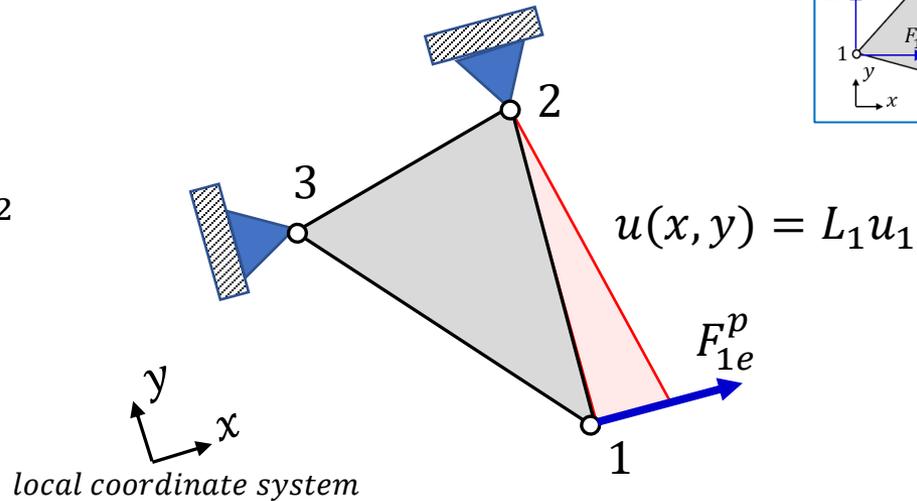
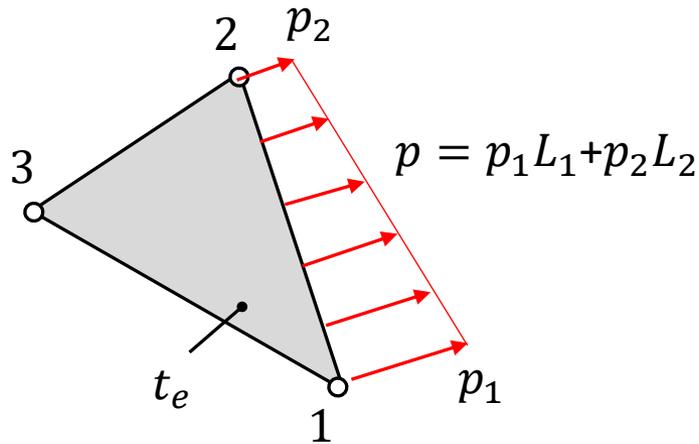
$$F_{1e}^p = t_e \int_0^l p(x,y) L_1 dy$$

Integration of barycentric functions:

$$J_1 = \int_0^l L_1^q L_2^r dl = \frac{q!r!l}{(q+r+1)!}$$

$$J_2 = \int_{A_e} L_1^q L_2^r L_3^t dA_e = \frac{q!r!t!}{(q+r+t+2)!} 2A_e$$

Example 2: Determination of the equivalent force in the CST element due to surface forces



The work of the equivalent force F_{1e}^p on displacement 1

The work of load $p(x,y)$ on displacement $u(x,y)$

$$F_{1e}^p \cdot 1 = t_e \int_0^l p(x,y) u(x,y) dy$$

$$F_{1e}^p = t_e \int_0^l (p_1 L_1 + p_2 L_2) L_1 dy$$

$$F_{1e}^p = t_e (p_1 \int_0^l L_1^2 dy + p_2 \int_0^l L_1 L_2 dy) = t_e (p_1 \frac{2!0!l}{(2+0+1)!} + p_2 \frac{1!1!l}{(1+1+1)!}) = t_e (\frac{1}{3} p_1 l + \frac{1}{6} p_2 l)$$

Integration of barycentric functions:

$$J_1 = \int_0^l L_1^q L_2^r dl = \frac{q!r!l}{(q+r+1)!}$$

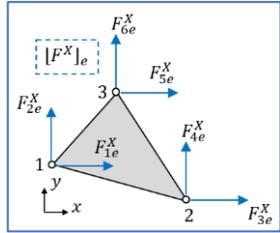
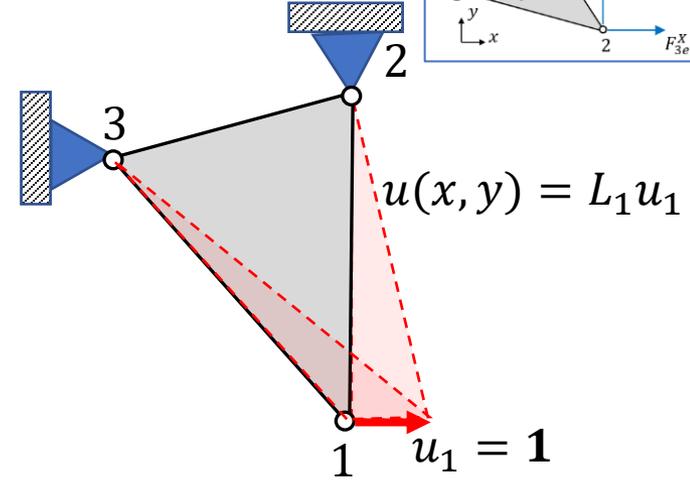
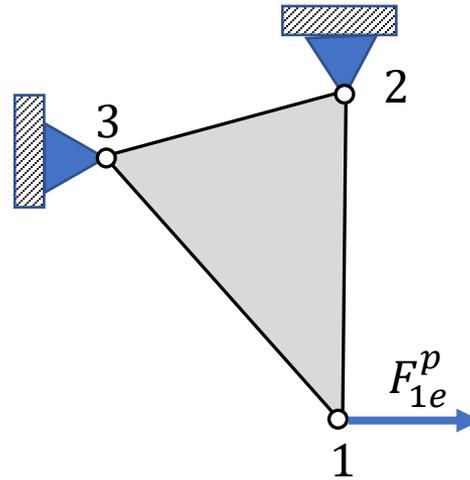
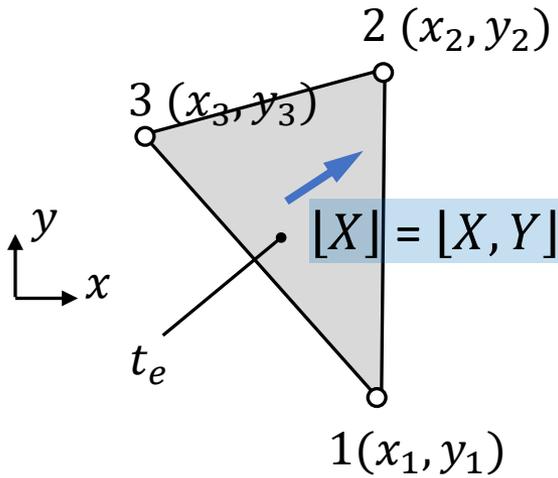
For $p_1 = p_2$:

$$F_{1e}^p = t_e \int_0^l p L_1 dy = t_e p \frac{1!0!l}{(1+0+1)!} = \frac{p l}{2} t_e$$

For $p_2 = 0$:

$$F_{1e}^p = t_e \int_0^l p_1 L_1^2 dy = t_e p_1 \frac{2!0!l}{(2+0+1)!} = \frac{p_1 l}{3} t_e$$

Example 3: Determination of the equivalent force in the CST element due to mass load



equivalent load vector due to mass load :

$$[F^X]_e = t_e \int_{A_e} [X][N] dA_e$$

The work of the equivalent force F_{1e}^X on displacement 1

$$F_{1e}^X \cdot 1 = t_e \int_{A_e} X u(x, y) dA_e$$

$$F_{2e}^X \cdot 1 = t_e \int_{A_e} Y v(x, y) dA_e$$

The work of load X on displacement $u(x, y)$

$$F_{1e}^X = t_e \int_{A_e} X L_1 dA_e$$

$$F_{2e}^X = t_e \int_{A_e} Y L_1 dA_e$$

Integration of barycentric functions:

$$J_2 = \int_{A_e} L_1^q L_2^r L_3^t dA_e = \frac{q!r!t!}{(q+r+t+2)!} 2A_e$$

For $X = const$ i $Y = 0$:

$$F_{1e}^X = t_e X \int_{A_e} L_1 dA_e = t_e X \frac{1!0!0!}{(1+0+0+2)!} 2A_e = X \frac{A_e t_e}{3}$$

$$F_{2e}^X = 0$$